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DISPERSION OF INTERNAL WAVES BY AN OBSTACLE FLOATING ON THE BOUNDARY SEPARATING TWO LIQUIDS*

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The problem of the scattering of a wave, that propagates along the boundary between two liquids, by a semi-infinite obstacle floating on this boundary is solved in a two-dimensional formulation. The solution is constructed using the Wiener-Hopf method interpreted by Jones in the framework of linear potential theory /1/. The fundamental properties of the processes of scattering and reflection of a wave by the obstacle are stated and an asymptotic analysis of the field in a far zone is presented.

1. We assume tht the half-space $z<0$ is filled with a homogeneous heavy incompressible liquid of density ρ_1, and the half-space $z>0$ is filled with a similar liquid of density ρ_2, where $\rho_1>\rho_2$. Suppose that there are massive particles of some substance floating on the surface $z=0$ between the liquids, and assume that the particles do not interact with each other as the separating boundary oscillates, or their interactions are negligible. The presence of such particles on the boundary between the liquids enables us to regard the boundary as a massive surface with a surface density of mass distribution $\sigma\geqslant 0$, where σ, being a function of the points of the boundary, may vanish in some of its regions.

We shall confine ourselves to the two-dimensional formulation, and we shall consider the case when the floating substance is contained in the half-plane $\{x>0,\ z=0\}$ only, and has constant density σ_0. The half-plane $\{x<0,\ z=0\}$ represents the free separating boundary. We shall ascribe the index 1 to all quantities related to the lower liquid, and the index 2 to those related to the upper liquid.

Suppose that a stationary internal wave of the form

$$u_j^\circ = u_j^\circ(x, z)\exp(-i\omega t) = (-1)^{j+1}A\exp(-a|z| + iax - i\omega t),\quad j=1,2$$
$$a = \omega^2(\rho_1+\rho_2)/[g(\rho_1-\rho_2)]$$

approaches the massive boundary from infinity along the boundary separating the liquids. Here $u_j^\circ\ (j=1,2)$ is the velocity potential and g is acceleration due to gravity.

We shall consider the problem of the diffraction of the internal wave u_j° on the massive part of the boundary. Let us express the amplitude U_j of the velocity potential as the sum

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of the potential of the incoming wave and the potential of the wave arising as a result of diffraction: $U_j = u_j^\circ(x, z) + u_j(x, z)$. For the potentials u_j describing the velocity field of the dispersed wave, the following boundary value problem for conjugation of harmonic functions arises:

$$\Delta u_j = 0, \quad j = 1, 2; \; z \neq 0 \tag{1.1}$$

$$\partial u_1/\partial z = \partial u_2/\partial z, \quad \forall x \in R^1, \quad z = 0$$

$$\omega^2(\rho_1 u_1 - \rho_2 u_2 + \sigma(x)\,\partial u_1/\partial z) - g(\rho_1 - \rho_2)\,\partial u_1/\partial z =$$
$$-aA\omega^2\sigma(x)\exp(iax), \quad z = 0 \quad |u_j| \to 0, \quad |z| \to \infty$$

$$\sigma(x) = \begin{cases} \sigma_0 = \text{const} > 0, & x > 0 \\ 0, & x < 0 \end{cases}$$

Moreover, the functions u_j should be bounded in a neighbourhood of the point $(x, z) = (0, 0)$ (the point of discontinuity of $\sigma(x)$) and their gradient should satisfy the estimate $|\nabla u_j| \leqslant C|\ln r|$ as $r = \sqrt{x^2 + z^2} \to 0$ ("conditions on the edge").

The dynamic condition (the third equality of (1.1)) can be derived on the basis of Newton's second law of motion for a massive surface and the Bernoulli integral (with this end in view, see /2/).

If the functions $u_j(x, z)$ $(j = 1, 2)$ are the solution of problem (1.1), the function $v(x, z) = u_1(x, z) + u_2(x, -z)$ defined for $z \leqslant 0$ satisfies the following conditions: v is a harmonic function for $z < 0$, $\partial v(x, 0)/\partial z = 0$ and $v(x, z) \to 0$ as $|z| \to \infty$. Hence, by the appropriate theorem on uniqueness for harmonic functions, it follows that $v(x, z) \equiv 0$ for $z < 0$, and so $u_2(x, z) = -u_1(x, -z)$ for $z > 0$. Consequently, the solution of problem (1.1) can be reduced to searching for a function $u_1(x, z)$ defined for $z \leqslant 0$ and being a solution to the problem below, which follows from (1.1):

$$\Delta u = 0, \quad (x, z) \in R_-^2 \equiv \{(x, z)\colon x \in R^1, \; z < 0\} \tag{1.2}$$
$$u_z - au = 0, \quad z = 0, \; x < 0$$
$$u_z - bu = Aa(b - a)\exp(iax), \quad z = 0, \; x > 0$$
$$|u| \to 0, \quad z \to -\infty$$
$$|u| \leqslant C, \quad |\nabla u| \leqslant C_1|\ln r|, \quad r \to 0$$
$$b = \omega^2(\rho_1 + \rho_2)/[\sigma_0(\omega_s^2 - \omega^2)], \quad \omega_s^2 = g(\rho_1 - \rho_2)/\sigma_0$$

Here and henceforth the index of u_1 is dropped.

2. One can construct the solution of problem (1.2) by reducing it to the Riemann-Hilbert problem with discontinuous coefficients /3, 4/. We shall use a somewhat different but essentially equivalent method. We shall construct the solution of problem (1.2) as the limit as $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0$; $\varepsilon_1, \varepsilon_2 > 0$ of the solutions $u_\varepsilon(x, z)$ of the equation $\Delta u_\varepsilon + \varepsilon^2 u_\varepsilon = 0$ satisfying all conditions in (1.2) with the function $\exp(iax)$ on the right-hand side of the second condition replaced by $\exp(ikx)$, where $k = \sqrt{a^2 + \varepsilon^2}$, and with the third condition replaced by the stronger condition $|u_\varepsilon| + |\nabla u_\varepsilon| \leqslant C\exp(-\delta(\varepsilon)r)$ as $r \to \infty$, where $\delta(\varepsilon) \to +0$ as $\varepsilon \to 0$. We shall refer to the problem of defining the function $u_\varepsilon(x, z)$ as the auxiliary problem.

The solution of the auxiliary problem can be constructed using Jones's interpretation of the Wiener-Hopf method /5/, and has the form

$$u_\varepsilon(x, z) = \frac{Ak(b - a)}{\pi i F(k)} \int \frac{\exp[\gamma(\alpha) z - i\alpha x]}{G(\alpha)(\alpha^2 - k^2)}\, d\alpha \tag{2.1}$$
$$F(k) = L_+(k), \quad G(\alpha) = (\gamma(\alpha) + |b|)/L_-(\alpha), \quad \omega > \omega_s$$
$$F(k) = -M_+(k)(k + \alpha_0), \quad G(\alpha) = (\alpha + \alpha_0)/[M_-(\alpha)(\gamma(\alpha) + b)],$$
$$\omega < \omega_s$$
$$F(k) = P_+(k)(a - b), \quad G(\alpha) = 1/P_-(\alpha), \quad \omega = \omega_s$$
$$\alpha_0 = \sqrt{b^2 + \varepsilon^2}$$

(for $\omega = \omega_s$, $b = \pm\infty$ and boundary condition (1.2) takes the form $u = -A\exp(ikx)$. Besides, the second condition on the edge should be replaced by the condition $|\nabla u| \leqslant Cr^{-1/2}$, $r \to 0$). Here and henceforth we integrate over the entire real axis.

Let us describe the notation used in formulae (2.1). We denote by $\gamma(\alpha)$ the branch of the function $\sqrt{\alpha^2 - \varepsilon^2}$ that takes the value $-i\varepsilon$ for $\alpha = 0$. To single out this branch in the α-plane, we make the following cuts: a vertical cut directed upwards starting at the point $\alpha = \varepsilon$, and another vertical cut directed downwards starting as $\alpha = -\varepsilon$. Thus, the branching points $\alpha = \pm\varepsilon$ are connected by a cut passing through an infinitely distant point.

The functions $L_{\pm}(\alpha)$ and $M_{\pm}(\alpha)$ in (2.1) are the factors of a decomposition of the following functions into products:

$$L(\alpha) = [\gamma(\alpha) + a][\gamma(\alpha) + |b|] = L_+(\alpha) L_-(\alpha) \qquad (2.2)$$
$$M(\alpha) = [\gamma(\alpha) + a]/[\gamma(\alpha) + b] = M_+(\alpha) M_-(\alpha)$$

such that $L_-(\alpha) = L_+(-\alpha)$, $M_-(\alpha) = M_+(-\alpha)$ and $L_+(\alpha) \sim C_1\alpha$, $M_+(\alpha) \sim C_2$ for $\operatorname{Im}(\alpha) > -\delta(\varepsilon)$, $|\alpha| \to \infty$.

The factorization of functions (2.2) can be obtained on the basis of the factorization of the function $P(\alpha) = \gamma(\alpha) + a$, $a > 0$ such that $P_-(\alpha) = P_+(-\alpha)$ /5/:

$$P_+(\alpha) = \sqrt{a - i\varepsilon}\exp\left\{\int_0^{\alpha}\left[\frac{\xi + k}{2} + \xi\Gamma_+(\xi) + k\Gamma_+(-k)\right]\frac{d\xi}{\xi^2 - k^2}\right\} \qquad (2.3)$$
$$\Gamma_+(\xi) = \frac{ia}{\pi\gamma(\xi)}\ln\frac{\gamma(\xi) - \xi + i\varepsilon}{\gamma(\xi) + \xi - i\varepsilon}$$

Here and below $\ln z = \ln|z| + i\arg z$, where $-\pi/2 \leqslant \arg z \leqslant 3\pi/2$.

To obtain the solution of problem (1.2), it is necessary to pass to the limit as $\varepsilon \to 0$ in formulae (2.1). To achieve this aim, we need, in particular, to find the limiting expression for the function (2.3) as $\varepsilon \to 0$, which turns out to be equal to

$$P_+(\alpha) = P_-(-\alpha) = \sqrt{\alpha + a}\exp\left\{\frac{1}{\pi i}\int_0^{\alpha/a}\ln\xi\,\frac{d\xi}{\xi^2 - 1}\right\} \qquad (2.4)$$

As a result of passing to the limit as $\varepsilon \to 0$, some poles on the real axis appear for the integrals in (2.1). Thus, to evaluate these integrals, one should pass round the singular points of the integrands in such a way as to pass above the negative poles and below the positive ones. Using formulae (2.2)-(2.4) in the integrals in (2.1), one should take into account that for $\varepsilon = 0$,

$$k = a, \quad \alpha_0 = b, \quad \gamma(\alpha) = \sqrt{\alpha^2} = \alpha\operatorname{sgn}\operatorname{Re}\alpha$$

It is easy to check that, after passing to the limit, formulae (2.1) give the solution of problem (1.2), which by means of simple algebraic reduction based on formulae (2.2)-(2.4) can be written in the following form:

$$u(x, z) = \frac{Aa(b - a)}{\pi iF(a)}\int\frac{\exp[\alpha z\operatorname{sgn}\operatorname{Re}\alpha - i\alpha x]}{G(\alpha)(\alpha\operatorname{sgn}\operatorname{Re}\alpha - a)}\,d\alpha \qquad (2.5)$$
$$G(\alpha) = \begin{cases} L_+(\alpha), & \omega > \omega_s \\ M_+(\alpha)(\alpha + b), & \omega < \omega_s \\ P_+(\alpha), & \omega = \omega_s \end{cases}$$

The multiplier F is defined in (2.1)

3. To analyse the complete wave field $U = u^\circ + u$ with $u^\circ = u_1^\circ$ as $r = \sqrt{x^2 + z^2} \to \infty$, we introduce the polar system of coordinates $x = r\cos\varphi$, $z = r\sin\varphi$, $-\pi \leqslant \varphi \leqslant 0$, and we replace the integrals in (2.5) by integrals over the bisectors of the first and second quadrants of the α-plane for $x < 0$, and over the bisectors of the third and fourth quadrants for $x > 0$. The integrands decrease exponentially on the bisectors, and we can use integration by parts to evaluate the integrals. Up to terms of order $O(r^{-2})$, asymptotic estimation of the integrals over the bisector gives

$$u_d = \frac{A(b - a)}{\pi iF(a)G(0)}\,\frac{\sin\varphi}{r} \qquad (3.1)$$

Transforming the contours of integration as described above, one has to take into account the residues at the simple poles $\alpha = \pm a$ and $\alpha = -b$. The poles $\alpha = -a$ and $\alpha = -b$ should be taken into account for $x > 0$. If $x > 0$, then the residue at the point $\alpha = -a$ cancels the incoming wave u° in the expression for the complete wave field, while the residue at the pole $\alpha = -b$, which is taken into account for $b > 0$ (or $\omega < \omega_s$) gives the surface wave

$$u_s = \frac{4AabM_+(b)}{(a + b)^2 M_+(a)}\exp(bz + ibx) \qquad (3.2)$$

It follows that for $x > 0$, the complete wave field can be written in the form

$$U = u_d + p(\omega) u_s \tag{3.3}$$

where $p(\omega) = 1$ for $\omega < \omega_s$ and $p(\omega) = 0$ for $\omega \geqslant \omega_s$.

For $x < 0$, only the residues at the pole $\alpha = a$ contributes to the complete wave field. This contribution describes the reflected wave

$$u_R = A\,\Lambda(a) \exp(az - iax) \tag{3.4}$$

$$\Lambda(a) = \begin{cases} -L_-(a)/L_+(a), & \omega > \omega_s \\ (a-b) M_-(a)/[(a+b) M_+(a)], & \omega < \omega_s \\ -P_-(a)/P_+(a), & \omega = \omega_s \end{cases}$$

Therefore, the complete wave field for $x < 0$ can be expressed as

$$U = u^\circ + u_R + u_d$$

4. Formulae (3.1)-(3.5) given above describe the behaviour of the solution of problem (1.2), which is shown above to be closely related to the solution of problem (1.1). Namely, the solution of problem (1.2), which is defined for $z \leqslant 0$, extended to the upper half-plane $z > 0$ in such a way that it becomes an odd function, is a solution of problem (1.1) concerning diffraction of the internal wave u_j°, $j = 1, 2$ by the massive part of the separating boundary. This enables us to write down the corresponding formulae for the complete field in (1.1).

According to (3.3), the equality

$$U^j = u_d^j(x, z) + p(\omega) u_s^j(x, z), \quad j = 1, 2 \tag{4.1}$$

holds for $x > 0$, where the function $p(\omega)$ was defined earlier, and u_d^j and u_s^j are odd functions defined in the whole xz-plane by extending the functions u_d and u_s to the region $z > 0$. For instance,

$$u_d^j = \begin{cases} u_d(x, z), & j = 1, \quad z < 0 \\ -u_d(x, -z), & j = 2, \quad z > 0 \end{cases}$$

For $x < 0$, on the basis of (3.5), we obtain

$$U_j = u_j^\circ + u_R^j + u_d^j, \; j = 1, 2 \tag{4.2}$$

where u_j° is the incoming wave defined in Sect.1, and u_R^j is the odd extension of the function $u_R(x, z)$ to the whole plane.

Formulae (4.1) and (4.2) enable us to describe the diffraction problems in question. The internal wave described by u_j° propagates along the free boundary and is scattered by the edge as it reaches the floating substance. As a result, a reflected internal wave u_R^j appears (see (4.2)), which propagates along the free boundary against the incoming wave in the negative direction of the x-axis towards infinity. Moreover, part of the energy of the incoming wave is spent in exciting pure diffraction waves described by the term u_d^j in (4.1) and (4.2) and representing a superposition of radial running waves of the form $A_\pm r^{-1} \exp(\pm i\varphi - i\omega t)$. One can check the last assertion by considering the explicit form (3.1) of the function u_d and representing $\sin\varphi$ as a linear combination of exponents.

It follows from formula (4.1) that when $\omega < \omega_s$ there is another internal wave arising in addition to the waves described above. In terms of the theory of scattering the additional wave can be called the transmitted wave. This internal wave propagates in the positive direction of the x-axis along the massive part of the separating boundary, and its wavelength (see the expression for u_s) is always less than that of the internal wave on the free separating boundary (since $a < b$), and decreases infinitely as $\omega \to \omega_s - 0$. Its amplitude also tends to zero as $\omega \to \omega_s - 0$, and the wave vanishes for $\omega \geqslant \omega_s$. Thus, $\omega = \omega_s$ is a kind of threshold frequency, since for frequencies exceeding this value propagation of internal waves along the massive part of the boundary turns out to be impossible. In other words, the transmission coefficient for waves penetrating into the half-plane $x > 0$, which contains the massive part of the separating boundary, turns out to be equal to zero for $\omega \geqslant \omega_s$.

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